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LINEAR THERMODYNAMIC SYSTEMS WITH MEMORY. II. NECESSARY AND SUFFICIENT CONDITIONS FOR APPLICABILITY OF THE SECOND LAW OF THERMODYNAMICS

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The paper is the second work in a series devoted to nonequilibrium thermodynamics of linear systems with memory. A theorem is proved that contains necessary and sufficient conditions which should be satisfied by constitutive equations for such systems in order to meet the second law of thermodynamics.

Introduction. In the first article of the series [1], the main statements and postulates of the theory of linear thermodynamic systems with memory were formulated, an analog of the Coleman-Owen theorem for systems of the type was presented, and several subsidiary results (Lemmas 1-4) were proved. For all the notations and definitions see the preceding article. Here we will refer to expressions, postulates, and lemmas from this article by the number 1 in square brackets.

Here we will prove our main result, the theorem containing necessary and sufficient conditions for applicability of the second law of thermodynamics in the form of postulate P1 [1].

Theorem 1. *The second principle in the form of the postulate P1 is fulfilled if and only if the generalized modulus* E *and the relaxational function* R *have the properties*

$$\mathbf{E} = \mathbf{E}^{\times}, \tag{1}$$

$$\langle \alpha^*, \overset{\circ}{\mathbf{R}}_F(\omega)\, \alpha \rangle \geq 0 \tag{2a}$$

for all $\omega \in R$, $\alpha \in S^*$.

The function $\overset{\circ}{\mathbf{R}}_F$ in (2a) is defined in terms of $\mathbf{R}$ in (1.22) [1] and (1.27) [1].

We should note that according to Lemma 1 [1] condition (2a) can be presented in a different form:

$$\langle \alpha^*, (\mathbf{R}_L(\mu + i\omega) + \mathbf{R}_L^{\times}(\mu - i\omega))\, \alpha \rangle \geq 0 \tag{2b}$$

for all $\omega \in R$, $\alpha \in S^*$.

Separating the real and imaginary parts in (2a), one can express (2a) in terms of real-valued quantities:

$$\langle \alpha, (\mathbf{R}_c(\omega) + \mathbf{R}_c^{\times}(\omega))\, \alpha \rangle + \langle \beta, (\mathbf{R}_c(\omega) + \mathbf{R}_c^{\times}(\omega))\, \beta \rangle + 2 \langle \beta, (\mathbf{R}_s(\omega) - \mathbf{R}_s^{\times}(\omega))\, \alpha \rangle \geq 0 \tag{2c}$$

for all $\omega \geq 0$; $\alpha, \beta \in S$.

Here $\mathbf{R}_L$, $\mathbf{R}_c$, $\mathbf{R}_s$, and $\mathbf{R}_F$ are defined according to (1.24)-(1.27) [1]. Similar properties of relaxational functions are already known in fluid mechanics and other branches of thermodynamics [1-6] as corollaries of the irreversibility principle. However, the main value of the result obtained consists in the proof of the fact that the given properties are not only the corollaries of the second principle, but also are sole corollaries of the principle, the latter result being obtained within a rather general context.

1. Proof of Necessity in Theorem 1. In order to prove necessity here, we must show that conditions (1) and (2) imposed on E and R stem from P1 [1].

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Let P1 [1] be fulfilled. Let us show first that the symmetry of the generalized modulus E, i.e., Eq. (1), follows from the postulate. Let us consider an arbitrary process h of duration T such that

$$h^i(T) = 0. \tag{1.1}$$

Let us denote by h_λ a process of duration T/λ which for $\lambda > 0$ is defined in terms of the given process h in the following manner:

$$h_\lambda(s) = \lambda h(\lambda s). \tag{1.2}$$

It is easy to notice that the following relationship for h_λ follows from (1.1):

$$h_\lambda^i(T/\lambda) = \int_0^{T/\lambda} \lambda h(\lambda s)\, ds = h^i(T) = 0. \tag{1.3}$$

Let us show now that for any $\delta > 0$ and any equilibrium state $\Lambda_0^+ = \{\varepsilon_0, 0^+\}$ there exists a $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$

$$\| \Lambda_0^+ - P_{h_\lambda}^{T/\lambda} \|_s < \delta. \tag{1.4}$$

is satisfied. Indeed, introducing the notation

$$M_h = \sup \{ |h(s)| \mid s \in [0, T] \} \tag{1.5}$$

and taking into account (1.1) we build an estimate

$$\| \Lambda_0^+ - P_{h_\lambda}^{T/\lambda} \|_s = \left(|\varepsilon^0 - \varepsilon_{h_\lambda}|^2 + \int_0^{T/\lambda} \left| h_\lambda \left(\frac{T}{\lambda} - s \right) \right|^2 \xi(s)\, ds \right)^{1/2} \leq$$

$$\leq \left(\left| \varepsilon^0 - \left(\varepsilon^0 + h_\lambda^i \left(\frac{T}{\lambda} \right) \right) \right|^2 + \lambda^2 \int_0^{T/\lambda} |h_\lambda (T - \lambda_s)|^2 \xi(s)\, ds \right)^{1/2} =$$

$$= \left(\lambda \int_0^T |h(T - s')|^2 \xi \left(\frac{s'}{\lambda} \right) ds' \right)^{1/2} \leq \sqrt{\lambda}\, M_h \xi(0)\, T, \tag{1.6}$$

whence the above-formulated statement follows. It follows from this statement and the assumption of fulfilment of P1 [1] that for any $\epsilon > 0$ there exists a $\lambda_0 > 0$ such that from $\lambda < \lambda_0$ for any state Λ_0^+ and process h_λ (where h is the earlier defined process) the following relationship follows:

$$a(\Lambda_0^+, h_\lambda) > -\epsilon. \tag{1.7}$$

Let us write the right side of this inequality in detail taking into account (2.1) [1], (1.11) [1], and (1.13)-(1.16) [1], and rearrange it by changing the integration order and substituting variables:

$$-\epsilon < \int_0^{T/\lambda} \langle \mathrm{E}(\varepsilon_0 + h_\lambda^i(\tau)),\ h_\lambda(\tau) \rangle\, dt + \int_0^{T/\lambda} \left\langle \left(\int_0^\tau \mathrm{R}(s)\, h_\lambda(\tau - s)\, ds \right),\ h_\lambda(\tau) \right\rangle d\tau =$$

$$= \int_0^{T/\lambda} \langle \mathrm{E}(\varepsilon_0 + h^i(\lambda\tau)),\ h(\lambda\tau) \rangle \lambda\, d\tau + \int_0^{T/\lambda} \left\langle \left(\int_0^\tau \mathrm{R}(s)\, h(\lambda(\tau - s))\, \lambda\, ds \right),\ h(\lambda\tau) \right\rangle \lambda\, d\tau =$$

$$= \int_0^T \left\langle \mathrm{E}\varepsilon_h(\tau),\ \frac{d}{d\tau}\varepsilon_h(\tau) \right\rangle d\tau + \int_0^T \int_s^T \left\langle \mathrm{R}\left(\frac{s}{\lambda} \right) h(\tau - s),\ h(\tau) \right\rangle d\tau\, ds. \tag{1.8}$$

Here ε_h is defined in accordance with (1.17) [1] and in view of (1.1) it has the property $\varepsilon_h(0) = \varepsilon_h(T)$.

It can be shown that the last term in (1.8) approaches zero when $\lambda \to 0$. Indeed, using the definitions (1.20) [1], (1.21) [1], and (1.5) we obtain

$$\left| \int_0^T \int_s^T \left\langle \mathbf{R}\left(\frac{s}{\lambda}\right) h(\tau - s),\ h(\tau) \right\rangle d\tau\, ds \right| \le$$

$$\le \int_0^T r\left(\frac{s}{\lambda}\right) \int_s^T |h(\tau - s)|\,|h(\tau)|\, d\tau\, ds \le M_h^2 T \left(\int_0^{\sqrt{\lambda}} r\left(\frac{s}{\lambda}\right) ds + \int_{\sqrt{\lambda}}^T r\left(\frac{s}{\lambda}\right) ds \right) \le$$

$$\le M_h^2 T\,(\sqrt{\lambda}\,\tilde{r}(0) + T\tilde{r}(1/\sqrt{\lambda}\,)). \tag{1.9}$$

The latter expression approaches zero when $\lambda \to 0$ since $\tilde{r}(\infty) = 0$. Because of this fact we obtain, by making the passage to the limit $\lambda \to 0$ in (1.8) and taking into account that $\epsilon > 0$ is arbitrary, we obtain

$$\int_0^T \left\langle \mathbf{E}\varepsilon_h(\tau),\ \frac{d}{d\tau}\varepsilon_h(\tau) \right\rangle d\tau \ge 0 \tag{1.10}$$

for any ε_h such that $\varepsilon_h(0) = \varepsilon_h(T)$. As is shown in [6, Chap. 1, para. 2] from this fact symmetry of E follows, i.e., $\mathbf{E} = \mathbf{E}^{\times}$.

Now we turn to the proof of the fact that the inequality (2a) follows from P1 [1]. We consider P1 [1] to be fulfilled, and for an arbitrary equilibrium state $\Lambda_0^+ = \{\varepsilon_0, 0^+\}$ we consider the process h_0 of duration $T_0 = T + T_1 + T_2$, which is a composition of an arbitrary process h of duration T, a process h_1 of duration T_1, and a stationary process u of duration T_2, with h_1 having a special form:

$$h_1(s) = -\frac{1}{T_1} h^i(T) = \text{const}, \tag{1.11}$$

where the notation of (1.16) [1] is used.

Thus,

$$h_0 = h \circ h_1 \circ u, \tag{1.12}$$

and, as is clearly seen,

$$h_0^i(T_0) = 0. \tag{1.13}$$

Let us consider the expression on the left side of (2.3) [1] for the specified state and process. Using the definitions of the norm, the process, the associated transformation, and the composition of processes ((1.5) [1], (1.7) [1], (1.13)-(1.16) [1]) this expression can be represented as follows:

$$\| \Lambda_0^+ - P_{h_0}^{T_0} \Lambda_0^+ \|_s = \left[|\varepsilon^0 - h_0^i(T_0)|^2 + \int_0^{T_2} \xi(s)^2\, |h_0^i(T_0)|^2\, ds + \right.$$

$$\left. + \int_{T_2}^{T_2+T} \xi(s) \left| \frac{h^i(T)}{T_1} \right|^2 ds + \int_{T_1+T_2}^{T_0} \xi(s)\, |h(T_0 - s)|^2\, ds \right]^{1/2}. \tag{1.14}$$

Using (1.13) and the fact that $\xi(\cdot)$ decreases monotonically we obtain the following upper estimate for (1.14):

$$\| \Lambda_0^+ - P_{h_0}^{T_0} \Lambda_0^+ \|_s = \left[\left| \frac{h^i(T)}{T_1} \right|^2 \int_{T_2}^{T_2+T} \xi(s)\, ds + \int_0^T |h(s)|^2\, \xi(T_0 - s)\, ds \right]^{1/2} \le$$

$$\leq \left[\frac{|h^i(t)|^2}{T_1} \xi(T_2) + (|h|^2)^i(T)\, \xi(T_2+T_1) \right]^{1/2} \leq \left[\frac{|h^i(T)|^2}{T_1} + (|h|^2)^i(T) \right]^{1/2} \sqrt{\xi(T_2)}\,. \quad (1.15)$$

Thus, it is clear that for arbitrary $\delta > 0$, $T_1 > 0$, $T > 0$, and $h \in \mathscr{P}$ one can choose a T_2 large enough to satisfy the condition

$$\left[\frac{|h^i(T)|^2}{T_1} + (|h|^2)^i(T) \right]^{1/2} \sqrt{\xi(T_2)} < \delta\,. \quad (1.16)$$

since $\lim_{T\to 0} \xi(T) = 0$, and the expression in the square brackets does not depend on T_2. Substituting (1.16) into (1.15) we find that for any $\delta > 0$ one can choose a T_2 such that condition (2.3) [1] is satisfied for the chosen state and process. It follows herefrom that, in view of the assumption of the applicability of P1 [1], for any $\epsilon > 0$ one can choose a T_2 such that for any h and T_1 the condition

$$a(\Lambda_0^+, h_0) > -\epsilon\,. \quad (1.17)$$

is satisfied. We write the right side of this inequality in detail using definitions (1.13)-(1.16) [1], (2.1) [1], and (1.12):

$$\int_0^{T_0} \langle \mathbf{E}(\varepsilon_0 + h_0^i(\tau)),\, h_0(\tau) \rangle\, d\tau + \int_0^T \int_0^\tau \langle \mathbf{R}(s)\, h(\tau - s),\, h(s) \rangle\, ds\, d\tau -$$

$$- \frac{1}{T_1} \int_T^{T+T_1} \left(\frac{1}{T_1} \int_0^{\tau-T} \langle \mathbf{R}(s)\, h^i(T),\, h^i(T) \rangle\, ds + \int_{\tau-T}^T \langle \mathbf{R}(s)\, h(\tau - s),\, h^i(T) \rangle\, ds \right) d\tau > -\epsilon\,. \quad (1.18)$$

We calculate here the first integral using the fact that $\mathbf{E} = \mathbf{E}^\times$ and change the variable in the first and last integrals:

$$\frac{1}{2} \langle \mathbf{E}(\varepsilon_0 + h_0^i(T_0)),\, \varepsilon_0 + h_0^i(T_0) \rangle - \frac{1}{2} \langle \mathbf{E}\varepsilon_0,\, \varepsilon_0 \rangle +$$

$$+ \int_0^T \int_0^\tau \langle \mathbf{R}(\tau - s)\, h(s),\, h(\tau) \rangle\, ds\, d\tau + \frac{1}{T_1^2} \int_T^{T+T_1} \int_0^{\tau-T} \langle \mathbf{R}(s)\, h^i(T),\, h^i(T) \rangle\, ds -$$

$$- \frac{1}{T_1} \int_T^{T+T_1} \int_{\tau-T}^T \langle \mathbf{R}(\tau - s)\, h(s),\, h^i(T) \rangle\, ds\, d\tau > -\epsilon\,. \quad (1.19)$$

It should be noted that in view of (1.13), the first two terms in (1.19) cancel out. In addition, an estimate can be built for two last terms in (1.19) using the properties of the function $\bar{r}$ defined in (1.21) [1]:

$$\Bigg| \frac{1}{T_1^2} \int_T^{T+T_1} \int_0^{\tau-T} \langle \mathbf{R}(s)\, h^i(T),\, h^i(T) \rangle\, ds -$$

$$- \frac{1}{T_1} \int_T^{T+T_1} \int_{\tau-T}^T \langle \mathbf{R}(\tau - s)\, h(s),\, h^i(T) \rangle\, ds\, d\tau \Bigg| \leq \frac{1}{T_1^2} |h^i(T)|^2 \int_T^{T+T_1} (\bar{r}(0) -$$

$$- \bar{r}(\tau - T))\, d\tau + \frac{|h^i(T)|}{T_1} \int_0^T (\bar{r}(T - s) - \bar{r}(T + T_1 - s))\, |h(s)|\, ds \leq$$

$$\leq \frac{1}{T_1^2} |h^i(T)|^2\, \bar{r}(0) = \frac{|h^i(T)|}{T_1} \int_0^T |h(s)|\, ds\,. \quad (1.20)$$

In view of these facts the inequality (1.19) is reduced to

$$\int_0^T \int_0^\tau \langle \mathrm{R}(\tau - s)\, h(s),\ h(\tau) \rangle \, ds\, d\tau > - \frac{|h^i(T)|}{T_1} \bar{r}(0) \left(|h^i(T)| + \int_0^T |h(s)|\, ds \right) - \epsilon . \tag{1.21}$$

Inasmuch as (1.21) should be satisfied for any T_1 and ϵ (including both arbitrary large T_1 and arbitrary small ϵ) it follows from this inequality that

$$\int_0^T \int_0^\tau \langle \mathrm{R}(\tau - s)\, h(s),\ h(\tau) \rangle \, ds\, d\tau \geq 0 . \tag{1.22}$$

Hence by virtue of Lemma 1 [1], the inequality being proved follows.

2. The Proof of Sufficiency in Theorem 1. In order to prove sufficiency one must, assuming that conditions (1) and (2) are satisfied, show that P1 [1] is satisfied in this case. For an arbitrary state $\Lambda = \{\alpha, f\} \in \mathcal{S}$ and any $\delta > 0$ we choose a process h of duration T such that the inequality

$$\| \Lambda - P_h^T \Lambda \|_s < \delta , \tag{2.1}$$

is satisfied, and show that if (1) and (2) hold, and, in view of Lemma 1 [1] and (3.8) [1], for any $\epsilon > 0$ in (2.1), a δ can be chosen such that for the state and process under consideration

$$a(\Lambda,\ h) > - \epsilon . \tag{2.2}$$

is satisfied.

Let us write (2.1) explicitly:

$$\left[|\alpha - (\alpha + h^i(T))|^2 + \int_0^T \xi(s)\, |f(s) - h(T - s)|^2\, ds + \right.$$

$$\left. + \int_T^\infty \xi(s)\, |f(s) - f(s - T)|^2\, ds \right]^{1/2} < \delta .$$

Hence

$$|h^i(T)|^2 < \delta^2 , \tag{2.3}$$

$$\int_0^T \xi(s)\, |f(s) - h(T - s)|^2\, ds < \delta^2 , \tag{2.4}$$

$$\int_T^\infty \xi(s)\, |f(s) - f(s - T)|^2\, ds < \delta^2 . \tag{2.5}$$

The left side of inequality (2.2), which must be proved, is as follows:

$$a(\Lambda,\ h) = \int_0^T \langle h(\tau),\ \left[\mathrm{E}(\alpha + h^i(\tau)) + \int_0^\tau \mathrm{R}(s)\, h(\tau - s)\, ds + \int_\tau^\infty \mathrm{R}(s)\, f(s - \tau)\, ds \right] \rangle \, d\tau . \tag{2.6}$$

Let us introduce the notation

$$g(s) \overset{\text{def}}{=} h(s) - f(T - s) . \tag{2.7}$$

In view of (2.7), inequality (2.4) can be written as follows:

$$\int_0^T \xi(s)\, |g(T-s)|^2\, ds < \delta^2 . \tag{2.8}$$

We express h in (2.6) in terms of g and f using (2.7), first integrating the first term in the square brackets and changing the variable in the integrals:

$$a(\Lambda, h) = \frac{1}{2}\Big(\langle (\alpha + h^i(T)), \mathrm{E}(\alpha + h^i(T)) \rangle - \langle \alpha, \mathrm{E}\alpha \rangle \Big) +$$

$$+ \int_0^T \int_0^\tau \langle f(T-\tau), \mathrm{R}(\tau - s) f(T-s) \rangle\, ds\, d\tau + \int_0^T \int_0^\tau \langle f(T-\tau), \mathrm{R}(\tau - s) g(s) \rangle\, ds\, d\tau +$$

$$+ \int_0^T \int_0^\infty \langle f(T-\tau), \mathrm{R}(\tau + s) f(s) \rangle\, ds\, d\tau + \int_0^T \int_0^\tau \langle g(\tau), \mathrm{R}(\tau - s) f(T-s) \rangle\, ds\, d\tau +$$

$$+ \int_0^T \int_0^\tau \langle g(\tau), \mathrm{R}(\tau - s) g(s) \rangle\, ds\, d\tau + \int_0^T \int_0^\infty \langle g(\tau), \mathrm{R}(\tau + s) f(s) \rangle\, ds\, d\tau \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} C + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 , \tag{2.9}$$

where notations introduced after the $\stackrel{\text{def}}{=}$ sign correspond to the order of the terms in expression (2.9).

Now, based on inequalities (2.3)-(2.5) (or (2.8) instead of (2.4)) and taking into account properties (1) and (2), we estimate the values of all the terms in (2.9).

In view of (2.3) we obtain for C:

$$|C| = \frac{1}{2} |\langle (\alpha + h^i(T)), \mathrm{E}(\alpha + h^i(T)) \rangle - \langle \alpha, \mathrm{E}\alpha \rangle| = |\langle h^i(T), \mathrm{E}\alpha \rangle +$$

$$+ \frac{1}{2} \langle h^i(T), \mathrm{E}h^i(T) \rangle| \le \|[\mathrm{E}]\|\, |\alpha|\, |h^i(T)| + \frac{1}{2} \|[\mathrm{E}]\|\, |h^i(T)|^2 < \|[\mathrm{E}]\| \left(|\alpha|\, \delta + \frac{1}{2}\delta^2 \right) . \tag{2.10}$$

Further, in obtaining estimates of integrals I_1 in (2.9) we take into account the fact that $f \in \mathcal{K}$ and, consequently, has limited support, i.e., for any f there exists a $T_f > 0$ such that $f(s) = 0$ for all $s > T_f$. We consider separately the cases of $T \le T_f$ and $T > T_f$ in (2.9).

A. $T \le T_f$. To obtain an estimate of the third integral I_3 in (2.9) we transform it as follows:

$$I_3 = \int_0^T \int_0^\infty \langle f(T-\tau), \mathrm{R}(\tau + s) f(s) \rangle\, ds\, d\tau = \int_0^T \int_0^T \langle f(T-\tau), \mathrm{R}(\tau + s) f(s) \rangle\, ds\, d\tau +$$

$$+ \int_0^T \int_T^\infty \langle f(T-\tau), \mathrm{R}(\tau + s) (f(s) - f(s-T)) \rangle\, ds\, d\tau +$$

$$+ \int_0^T \int_T^\infty \langle f(T-\tau), \mathrm{R}(\tau + s + T) f(s) \rangle\, ds\, d\tau . \tag{2.11}$$

The last of the integrals has a structure similar to that of integral I_3, except that the function $\mathrm{R}(\cdot + T)$ stands in the integral instead of $\mathrm{R}(\cdot)$; therefore a transformation similar to that used in (2.11) can be applied to the integral. As a result, it will be represented as a sum of three integrals to the last of which transformation (2.11) can be again applied, etc. Repeating this procedure $N + 1$ times we obtain

$$I_3 = \sum_{k=0}^{N} \int_0^T \int_0^T \langle f(T-\tau), \mathrm{R}(\tau + s + kT) f(s) \rangle\, ds\, d\tau +$$

$$+\int_0^T\int_T^\infty \langle f(T-\tau), \sum_{k=0}^{N} \mathbf{R}(\tau+s+kT)(f(s)-f(s-T))\rangle\, ds\, d\tau +$$

$$+\int_0^T\int_T^\infty \langle f(T-\tau), \mathbf{R}(\tau+s+(N+1)T)(f(s)-f(s-T))\rangle\, ds\, d\tau . \tag{2.12}$$

Inasmuch as the function $\mathbf{R}$ is absolutely integrable on R^+, one can pass to the limit $N\to\infty$ since the series in (2.12) converge. In addition, since $\mathbf{R}(\infty)=0$ the second term in (2.12) vanishes in this case. Passing to the limit and performing substitution of the variable in the integrals of the first sum in (2.12) we obtain

$$I_3=\sum_{k=0}^{\infty}\int_0^T\int_0^T \langle f(T-\tau), \mathbf{R}(\tau-s+kT)\,f(T-s)\rangle\, ds\, d\tau +$$

$$+\int_0^T\int_T^\infty \langle f(T-\tau), \sum_{k=0}^{N} \mathbf{R}(\tau+s+kT)(f(s)-f(s-T))\rangle\, ds\, d\tau \overset{\text{def}}{=} I_3'+I_3'' . \tag{2.13}$$

The notation I_3', I_3'' here corresponds to the order of the terms.

It should be noted that in view of Lemma 2 [1] from (2) which was assumed to be satisfied the inequality

$$I_1+I_3'\geq 0 . \tag{2.14}$$

follows.

Let us estimate the sum entering into integral I_3'' using the functions $\tilde{r}$ and $\bar{\tilde{r}}$ defined in (1.20) [1] and (1.23) [1]:

$$\left\|\left[\sum_{k=0}^{N}\mathbf{R}(\tau+s+kT)\right]\right\| \leq \sum_{k=0}^{N}\tilde{r}(\tau+s+kT)\leq \tilde{r}(\tau+s)+\frac{1}{T}\int_0^T \tilde{r}(\tau+s+\lambda)\, d\lambda =$$

$$=\tilde{r}(\tau+s)+\frac{1}{T}\bar{\tilde{r}}(\tau+s) . \tag{2.15}$$

The validity of the estimate follows from the definition and monotonicity of the function $\tilde{r}$.

With the use of (2.15) and the Cauchy-Schwarz-Bunyakovskii inequality the following estimate of the integral I_3 can be built:

$$|I_3''|\leq\int_0^T\int_T^\infty |f(T-\tau)|\left\|\left[\sum_{k=0}^{N}\mathbf{R}(\tau+s+kT)\right]\right\| |f(s)-f(s-T)|\, ds\, d\tau\leq$$

$$\leq\int_0^T\int_T^\infty |f(T-\tau)|\, \tilde{r}(\tau+s)+\frac{1}{T}\bar{\tilde{r}}(\tau+s))|f(s)-f(s-T)|\, ds\, d\tau\leq$$

$$\leq\left(\int_0^T\int_T^\infty |f(T-\tau)|^2\left(\tilde{r}(\tau+s)+\frac{1}{T}\bar{\tilde{r}}(\tau+s)\right) ds\, d\tau\right)^{1/2}\times$$

$$\times\left(\int_0^T\int_T^\infty \left(\tilde{r}(\tau+s)+\frac{1}{T}\bar{\tilde{r}}(\tau+s)\right)|f(s)-f(s-T)|^2\, ds\, d\tau\right)^{1/2} . \tag{2.16}$$

We introduce the following notation:

$$M_f\overset{\text{def}}{=}\sup\left\{\, |f(s)|\mid s\in R^+\right\} , \tag{2.17}$$

$$\mathcal{M} \overset{\text{def}}{=} \Big((\bar{\bar{r}}(0)\, T_f + \bar{\bar{\bar{r}}}(0))\, (M_1 T_f + M) \Big)^{1/2}, \tag{2.18}$$

where $\bar{\bar{\bar{r}}}$ is defined by analogy with (1.23) [1], and M_1 and M are the constants defined in (2.12) [1] and (2.5) [1].

Using this notation and considering the monotonicity of the functions $\bar{r}$ and $\bar{\bar{r}}$ and inequalities (2.5) and (2.13), as well as the condition $T \le T_f$, we can continue estimate (2.16):

$$|I_3''| \le \left(M_f^2 \int_0^T \left(\bar{r}(\tau + s) + \frac{1}{T}\, \bar{\bar{r}}(\tau + s) \right) ds\, d\tau \right)^{1/2} \times$$

$$\times \left(\int_T^\infty (\bar{r}(s)\, T + \bar{\bar{r}}(s))\, |f(s) - f(s - T)|^2\, ds \right)^{1/2} \le$$

$$\le \left(M_f^2 T \left(\bar{r}(0) + \frac{1}{T}\, \bar{\bar{r}}(0) \right) \right)^{1/2} \left(\int_T^\infty (M_1 T + M)\, |f(s) - f(s - T)|^2\, ds \right)^{1/2} \le$$

$$\le M_f \left((\bar{\bar{r}}(0)\, T_f + \bar{\bar{\bar{r}}}(0))\, (M_1 T_f + M)\, \delta^2 \right)^{1/2} = M_f \mathcal{M} \delta. \tag{2.19}$$

Let us estimate one of the three integrals I_2, I_4, and I_6 in (2.9). Taking into account the inequality

$$\int_0^T |g(T - s)|^2\, ds < \frac{\delta^2}{\xi(T)}, \tag{2.20}$$

which follows from (2.4), the notation (1.17) [1] and (1.20)-(1.23) [1], and the Cauchy-Schwarz-Bunyakovskii inequality we obtain for the integral I_2:

$$|I_2| \le \left(\int_0^T \int_0^\tau |f(T - \tau)|^2\, |[\mathbf{R}](\tau - s)]|\, ds\, d\tau \right)^{1/2} \times$$

$$\times \left(\int_0^T \int_0^\tau |g(s)|^2\, |[\mathbf{R}](\tau - s)]|\, ds\, d\tau \right)^{1/2} \le$$

$$\le \left(\int_0^T |f(T - \tau)|^2\, (\bar{r}(0) - \bar{r}(\tau))\, d\tau \right)^{1/2} \left(\int_0^T |g(s)|^2\, (\bar{r}(0) - \bar{r}(T - s))\, ds \right)^{1/2} \le$$

$$\le \left(M_f^2\, \bar{r}(0)\, T \right)^{1/2} \left(\bar{r}(0)\, \frac{\delta^2}{\xi(T)} \right)^{1/2} \le M_f \bar{r}(0) \sqrt{\left(\frac{T_f}{\xi(T_f)} \right)}\, \delta.$$

Similar calculations permit similar estimates for integrals I_4 and I_6, so that for the sum of these three integrals we obtain

$$|I_2| + |I_4| + |I_6| \le 3 M_f \bar{r}(0) \sqrt{\left(\frac{T_f}{\xi(T_f)} \right)}\, \delta. \tag{2.21}$$

Finally, for integral I_5 in (2.9), which remains untreated, we obtain in view of (2) and Lemma 1 [1]:

$$I_5 = \int_0^T \int_0^\tau \langle g(\tau),\ \mathbf{R}(\tau - s)\, g(s) \rangle\, ds\, d\tau \ge 0. \tag{2.22}$$

Combining estimates (2.10), (2.14), (2.19), (2.21), and (2.22) in (2.9) we obtain a final estimate of the action of the process h under consideration for the case of $T \le T_f$.

$$a(\Lambda, h) \geq -\,|[\mathbf{E}]|\,|\alpha|\,\delta - \frac{1}{2}\,|[\mathbf{E}]|\,\delta^2 - M_f \mathscr{M} \delta - 3M_f \bar{r}(0) \sqrt{\left(\frac{T_f}{\xi(T_f)}\right)}\,\delta. \tag{2.23}$$

B. We now consider the opposite case; namely, we let

$$T \geq T_f. \tag{2.24}$$

The boundedness of the support of the function f within the interval $[0, T_f]$ and condition (2.24) allow one to represent now inequality (2.5) in the form

$$\int_T^{T+T_f} \xi(s)\,|f(s-T)|^2\,ds < \delta^2. \tag{2.25}$$

Hence it follows that

$$\xi(T+T_f) \leq \delta^2 \left(\int_T^{T+T_f} |f(s-T)|^2\,ds\right)^{-1} = \frac{\delta^2}{I_f}, \tag{2.26}$$

where

$$I_f = \int_0^{T_f} |f(s)|^2\,ds. \tag{2.27}$$

We start with the derivation of an estimate of integral I_3 in (2.9), and to do this we use the inequalities (2.25) and (2.26) and Lemma 4 [1]:

$$|I_3| = \left| \int_{T-T_f}^{T} \int_0^{T_f} \langle f(T-\tau),\, \mathbf{R}(\tau+s) f(s) \rangle\,ds\,d\tau \right| \leq \int_{T-T_f}^{T} |f(T-\tau)|\,\bar{r}(\tau) M_f\,d\tau \leq$$

$$\leq M_f^2 \bar{r}(T-T_f)\,T_f \leq M_f^2 T_f M_{(2T_f)}\,\xi(T+T_f) < \frac{M_f^2 T_f M_{(2T_f)}}{I_f}\,\delta^2. \tag{2.28}$$

Here $M_{(2T_f)}$ is the constant introduced in Lemma 4 [1].

Then, in view of (2.24) we estimate the sum of the integrals $I_2 + I_4$ from (2.9):

$$|I_2 + I_4| \leq \left| \int_{T-T_f}^{T} \int_0^{\tau} \langle f(T-\tau),\, \mathbf{R}(\tau-s)\,g(s) \rangle\,ds\,d\tau \right| +$$

$$+ \left| \int_{T-T_f}^{T} \int_{T-T_f}^{\tau} \langle g(\tau),\, \mathbf{R}(\tau-s) f(T-s) \rangle\,ds\,d\tau \right|. \tag{2.29}$$

Using the Cauchy-Schwarz-Bunyakovskii inequality and changing the integration order where necessary, we obtain a continuation of estimate (2.29):

$$|I_2 + I_4| \leq \left(\int_{T-T_f}^{T} \int_0^{\tau} |f(T-\tau)|^2\,|[\mathbf{R}(\tau-s)]|\,ds\,d\tau \right)^{1/2} \times$$

$$\times \left(\int_{T-T_f}^{T} \int_0^{T} |g(s)|^2\,|[\mathbf{R}(\tau-s)]|\,ds\,d\tau + \int_0^{T-T_f} \int_{T-T_f}^{T} |g(s)|^2\,|[\mathbf{R}(\tau-s)]|\,ds\,d\tau \right)^{1/2} +$$

$$+\left(\int_{T-T_f}^{T}\int_{T-T_f}^{\tau} |f(T-\tau)|^2 \, |[\mathbf{R}(\tau-s)]| \, ds\, d\tau\right)^{1/2} \times$$

$$\times \left(\int_{T-T_f}^{T}\int_{T-T_f}^{\tau} |g(\tau)|^2 \, |[\mathbf{R}(\tau-s)]| \, ds\, d\tau\right)^{1/2} \le \left(M_f^2 T_f \bar{r}(0)\right)^{1/2} \times$$

$$\times \left(\bar{r}(0)\int_{T-T_f}^{T} |g(s)|^2 ds + \int_0^{T-T_f} |g(s)|^2 \bar{r}(T-T_f-s)\, ds\right)^{1/2} +$$

$$+\left(M_f^2 T_f \bar{r}(0)\right)^{1/2} \left(\int_{T-T_f}^{T} |g(\tau)|^2 d\tau\right)^{1/2}. \tag{2.30}$$

In order to continue this estimate we need the following inequalities, which follow from (2.8) and Lemma 4 [1]:

$$\delta^2 > \int_0^T \xi(s)\, |g(T-s)|^2 ds > \int_0^{T_f} \xi(s)\, |g(T-s)|^2 ds \ge \xi(T_f) \int_{T-T_f}^{T} |g(s)|^2 ds, \tag{2.31}$$

$$\int_0^{T-T_f} |g(s)|^2 \bar{r}(T-T_f-s)\, ds \le M_{(T_f)} \int_0^T |g(s)|^2 \xi(T-s)\, ds =$$

$$= M_{(T_f)} \int_0^T |g(T-s)|^2 \xi(s)\, ds < M_{(T_f)} \delta^2. \tag{2.32}$$

Using these inequalities we obtain

$$|I_2 + I_4| \le \left(M_f^2 T_f \bar{r}(0)\right)^{1/2} \left(\left(\frac{\bar{r}(0)}{\xi(T_f)} + \bar{r}(0) M_{(T_f)}\right)^{1/2} + \left(\frac{\bar{r}(0)}{\xi(T_f)}\right)^{1/2}\right)\delta =$$

$$= M_f \bar{r}(0) \sqrt{T_f} \left(\left(\frac{1}{\xi(T_f)} + M_{(T_f)}\right)^{1/2} + \left(\frac{1}{\xi(T_f)}\right)^{1/2}\right)\delta. \tag{2.33}$$

For further derivation it is convenient to introduce the function Ψ: $[0, 2T] \to S$ and to define it as follows:

$$\psi(s) = \begin{cases} f(T-\vec{s}) & \text{for} \quad s \in [0, T], \\ g(s-T) & \text{for} \quad s \in [0, 2T]. \end{cases} \tag{2.34}$$

Using this definition and changing integration variables we obtain for the sum of the three untreated integrals in (2.9):

$$I_1 + I_5 + I_6 = \int_0^{2T}\int_0^{\tau} \langle \psi(\tau), \mathbf{R}(\tau-s)\psi(T)\rangle \, ds\, d\tau \ge 0. \tag{2.35}$$

Here the inequality sign follows from Lemma 1 [1] since (2) was assumed to be satisfied. Summing estimates (2.10), (2.28), (2.33), and (2.35) we obtain the following estimate of expression (2.9) in the case under consideration:

$$a(\Lambda, h) \ge -\, |[\mathbf{E}]| \left(|\alpha|\,\delta + \frac{1}{2}\delta^2\right) - \frac{M_f^2 T_f M_{(2T_f)}}{I_f}\delta^2 -$$

$$- M_f \bar{r}(0) \sqrt{T_f} \left(\left(\frac{1}{\xi(T_f)} + M_{(T_f)} \right)^{1/2} + \left(\frac{1}{\xi(T_f)} \right)^{1/2} \right) \delta . \quad (2.36)$$

Combining inequality (2.23), which holds for $T \le T_f$ and (2.35) proved for $T > T_f$, we obtain an estimate of the action a that holds for all T:

$$a(\Lambda, h) \ge \min(a, b), \quad (2.37)$$

where a and b are the right sides of inequalities (2.23) and (2.36), respectively. In view of the obvious fact that if $a < 0$ and $b < 0$ then $\min(a, b) \ge a + b$, inequality (2.37) can be transformed to

$$a(\Lambda, h) \ge - A\delta^2 - B\delta, \quad (2.38)$$

where

$$A = |[\mathbf{E}]| + \frac{M_f^2 T_f M_{(2T_f)}}{I_f}, \quad (2.39)$$

$$B = 2|[\mathbf{E}]| \, |\alpha| + M_f \mathcal{M} + M_f \bar{r}(0) \sqrt{T_f} \left(\left(\frac{1}{\xi(T_f)} + M_{(T_f)} \right)^{1/2} + \left(\frac{1}{\xi(T_f)} \right)^{1/2} \right). \quad (2.40)$$

Recall that, as follows from the preceding consideration, inequality (2.38) holds if the process h satisfies constraints (2.1). It is obvious that if for an arbitrary $\epsilon > 0$ we choose a δ in (2.1) such that

$$0 < \delta < \frac{\sqrt{B^2 + 4A} - B^2}{2A},$$

is satisfied, then $-A\delta^2 - B\delta > - \epsilon$ and, consequently, (2.2) follows from (2.38) for the state and the process under consideration, Q.E.D.

Thus, the proof of the Theorem 1 is accomplished.

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REFERENCES

1. V. L. Kolpashchikov and A. I. Shnip, Inzh.-Fiz. Zh., **68**, No. 4, 633-646 (1995).
2. V. L. Kolpashchikov and A. I. Shnip, Int. J. Eng. Sci., **16**, No. 5, 503-514 (1978).
3. V. L. Kolpashchikov and A. I. Shnip, Inzh.-Fiz. Zh., **65**, No. 2, 192-197 (1993).
4. V. L. Kolpashchikov and A. I. Shnip, In: Theoretical and Applied Mechanics: The 4th National Congress (Varna 1981), Abstracts, Book 1, Sofia (1981).
5. M. E. Gurtin and I. Herrera, Quart. Appl. Math., **23**, 235-243 (1965).
6. W. A. Day, Thermodynamics of Simple Media with Memory [Russian translation], Moscow (1974).